Nonlinear Dynamics of Self-Sustained Oscillators

Building Advanced Technical Competencies Based on Fundamental Physics

Randall Tagg & Masoud Asadi-Zeydabadi
University of Colorado Denver
Students
Nhat Huang, Courtney Fleming, Tyler Sanford

Supported in part by NSF grant 1624882
as a part of the APS PIPELINE project https://epic.aps.org .

There is a missing piece in the standard repertoire for "vibrations & waves".

- Free oscillation of damped linear oscillators
  - Mass on a spring
  - Small-amplitude pendulum
  - RLC circuits
- Forced linear oscillators and resonance
- Nonlinear restoring force
  - Duffing oscillator
  - Large amplitude pendulum
- Chaos in forced systems

The Repertoire

but seldom any mention of

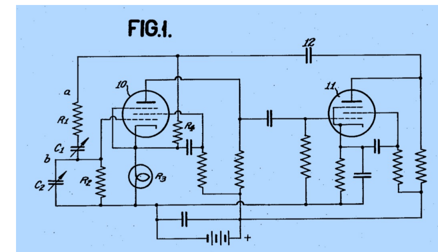
- Self-Sustained Oscillation

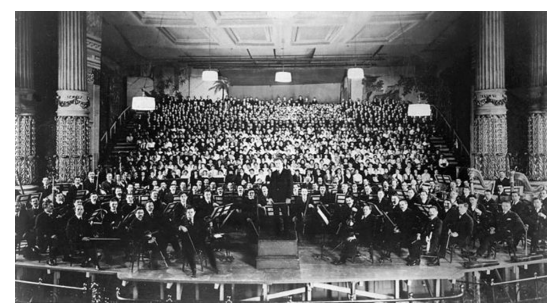
The design and use of oscillators is important to many technologies and innovations.

- **Value conceived**: a lower-cost stable audio oscillator: Wien-Bridge oscillator stabilized with a light bulb.
  (Hewlett's master's thesis).

- **Value created**: the HP 200A, Hewlett-Packard's first successful product, U.S. patent 2,268,872 (1939).

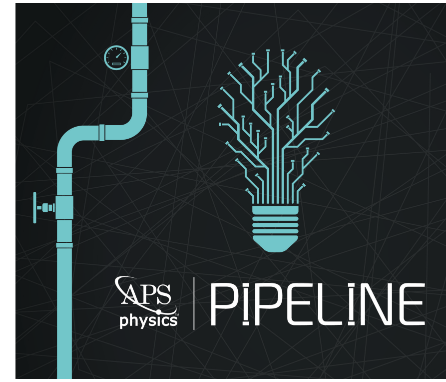
- **Value conveyed**: 8 units sold to Disney to test recording equipment during production of the film "Fantasia".





Images from Wikipedia

APS PIPELINE Network



- Six member institutions: Loyola University Maryland, Rochester Institute of Technology, Wright State, UC Denver, George Washington University, and William & Mary.

- Advised by experts from established physics entrepreneurship programs (e.g. Carthage College, Case Western, Kettering University)

- Goals are: → Physics, Innovation & Entrepreneurship
  - to **deliver tested PIE curriculum** to a wider cohort of practitioners.
  - to **assess of effects of PIE implementation** on student and faculty attitudes towards innovation and entrepreneurship, and **examine barriers** to PIE implementation
  - to **build a community** of expert practitioners who can mentor other institutions.

- Activities are varied in scope and resources needed; institutions varied in culture and resources available. For more information and resources, see:

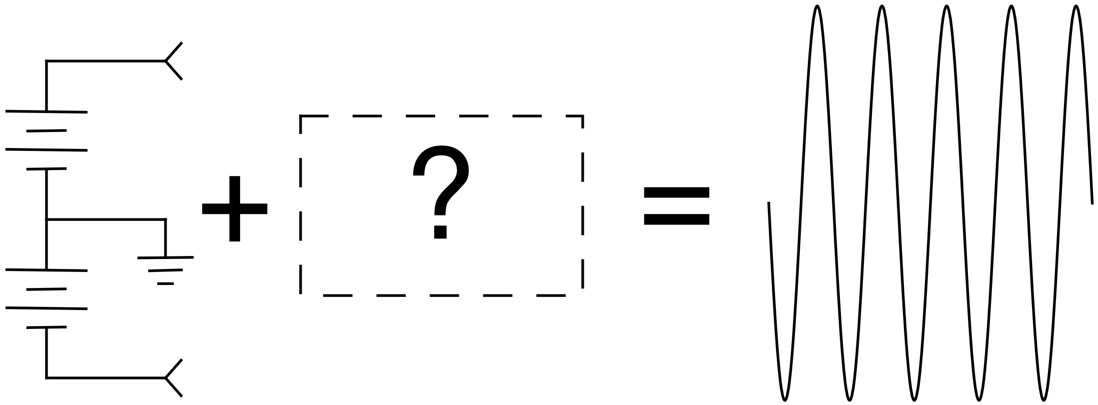
https://epic.aps.org

 bailey@aps.org

Support for this work was provided by the National Science Foundation's Improving Undergraduate STEM Education (IUSE) program under Award No. 1624882

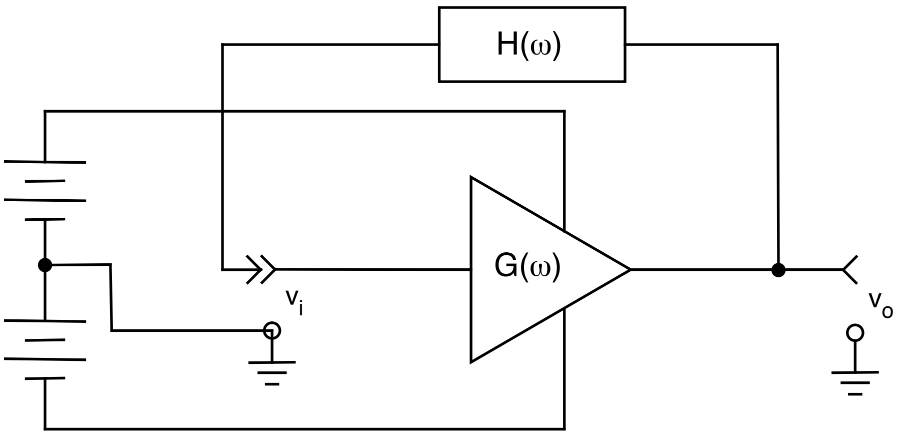
The central question:

What converts a steady power source into an oscillating signal?



A Necessary Condition for Sustained Oscillation

Barkhausen criterion



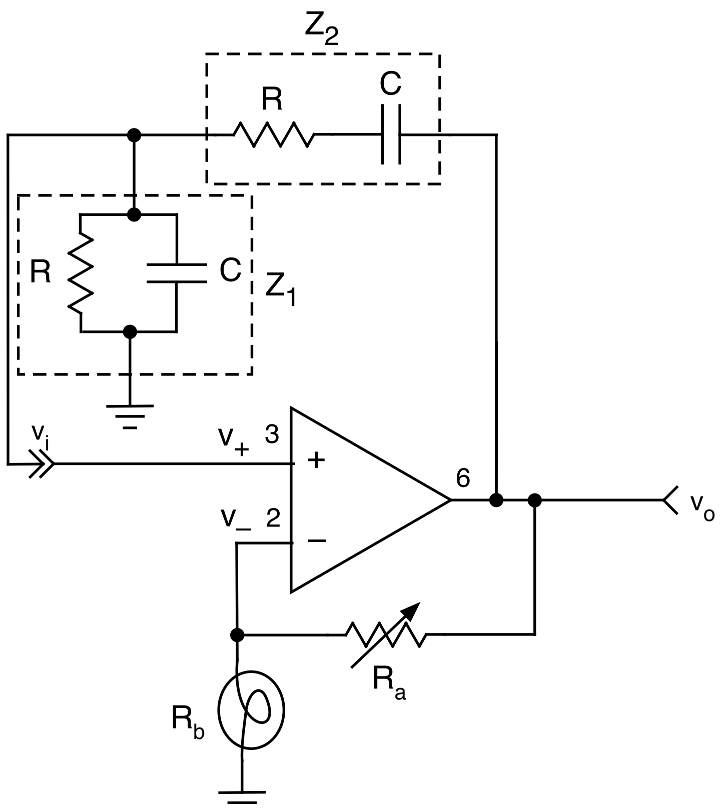
$$\widehat{V}_o = G(\omega)\widehat{V}_i.$$

$$\widehat{V}_i = H(\omega)\widehat{V}_o.$$

→ $\widehat{V}_i = G(\omega)H(\omega)\widehat{V}_i.$ → $G(\omega)H(\omega) = 1.$

A self-consistency criterion.

Wien Bridge oscillator



$$G \equiv 1 + \frac{R_a}{R_b}.$$

$$H(\omega) = \frac{1}{3 + j\left(\omega RC - \frac{1}{\omega RC}\right)}.$$

↓

$$G(\omega)H(\omega) = 1.$$

$$\frac{G}{3 + j\left(\omega RC - \frac{1}{\omega RC}\right)} = 1.$$

↓

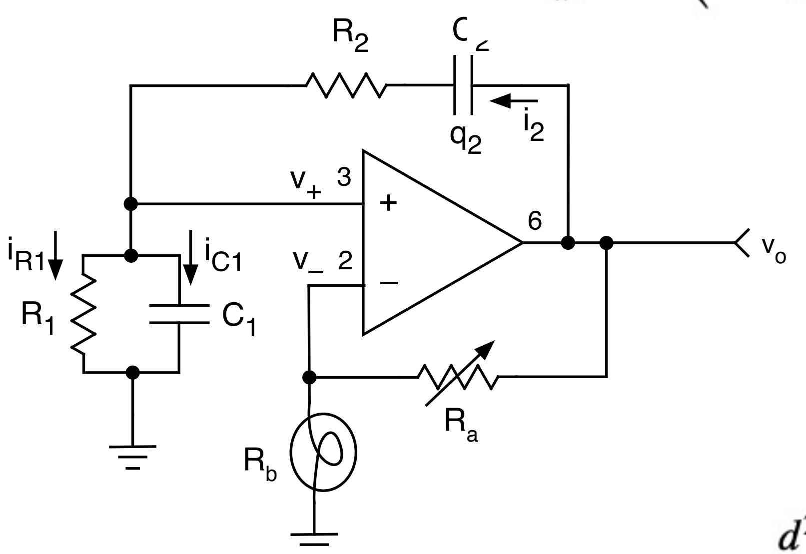
$$G = 3. \qquad \frac{R_a}{R_b} = 2.$$

$$\omega = \frac{1}{RC} \equiv \omega_0.$$

Finding a Sufficient Condition for Sustained Oscillation

Time-domain analysis

$$\frac{d^2 v_o}{dt^2} + \left(1 + \frac{R_2}{R_1} + \frac{C_1}{C_2} - G\right)\frac{1}{R_2 C_1}\frac{dv_o}{dt} + \frac{1}{(R_1 C_2)(R_2 C_1)} v_o = 0.$$



$$R_1 = R_2 \equiv R$$

$$C_1 = C_2 \equiv C$$

$$\frac{d^2 v_o}{dt^2} + (3 - G)\frac{1}{RC}\frac{dv_o}{dt} + \frac{1}{(RC)^2} v_o = 0.$$

$$G \equiv 1 + \frac{R_a}{R_b}.$$

This is the damped harmonic oscillator equation with the possibility of "negative damping" when G > 3.

Equivalent system of 1st order differential equations

$$\frac{dv}{dt} = \omega_0 w.$$

$$\frac{dw}{dt} = -\omega_0^2 v + \left(\frac{R_a}{R_b} - 2\right)\omega_0 w.$$

Time series and phase portraits

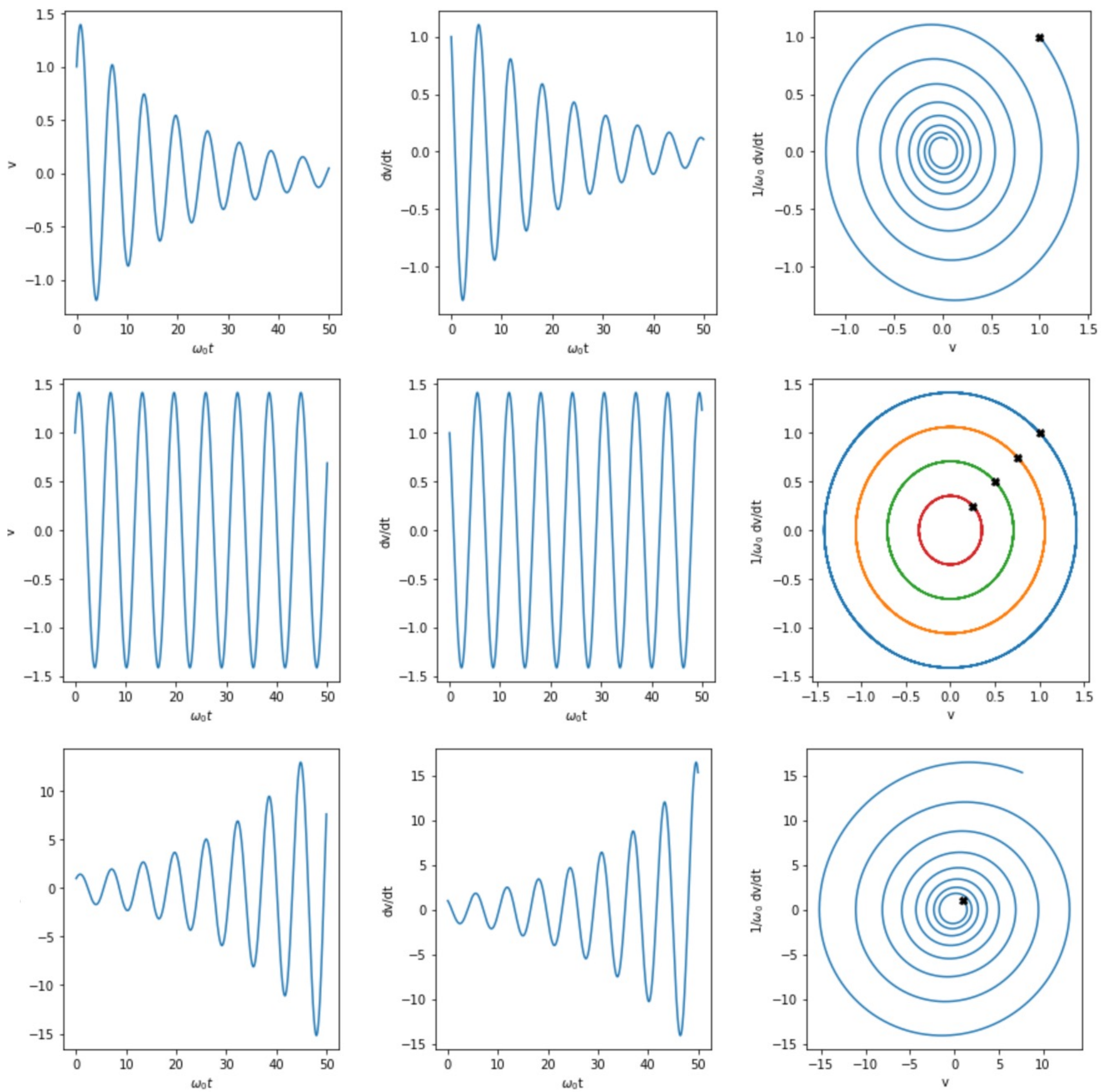
Define a bifurcation parameter $\varepsilon = R_a/R_b - 2$

v vs t | w vs t | w vs v

$\varepsilon = -0.1$

$\varepsilon = 0.$

$\varepsilon = +0.1$



Introduce nonlinearity through self-heating of the tungsten filament R_b, thus reducing the gain $R_a/R_b + 1$

$$\frac{dv}{dt} = \omega_0 w.$$

$$\frac{dw}{dt} = -\omega_0 v + \left(\frac{R_a}{R_{b0}\left[1 + \alpha(T - T_0) + \beta(T - T_0)^2\right]} - 2 \right) \omega_0 w.$$

$$\frac{d(T - T_0)}{dt} = \frac{1}{\mathbf{C}_f} \left[\frac{R_{b0}[1 + \alpha(T - T_0) + \beta(T - T_0)^2]}{\{R_a + R_{b0}[1 + \alpha(T - T_0) + \beta(T - T_0)^2]\}^2} v^2 - \mathrm{K}(T - T_0) - \mathrm{S}(T^4 - T_0^4) \right].$$

Heat capacity

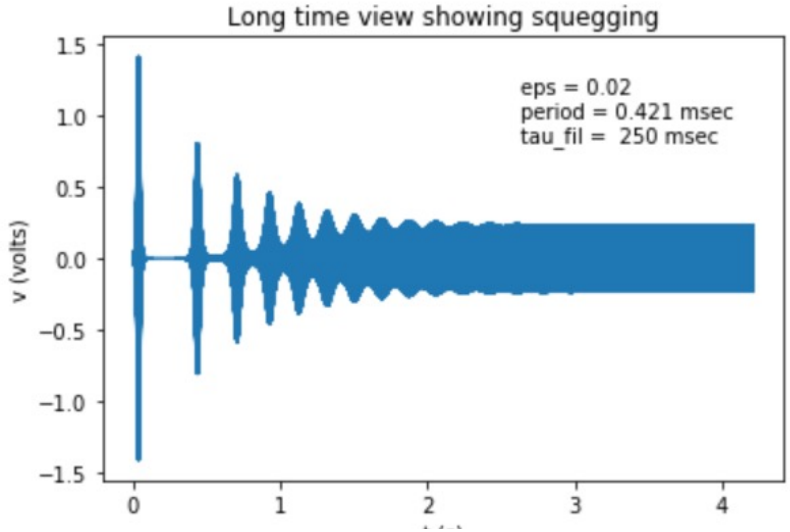
Power dissipated

Thermal conduction

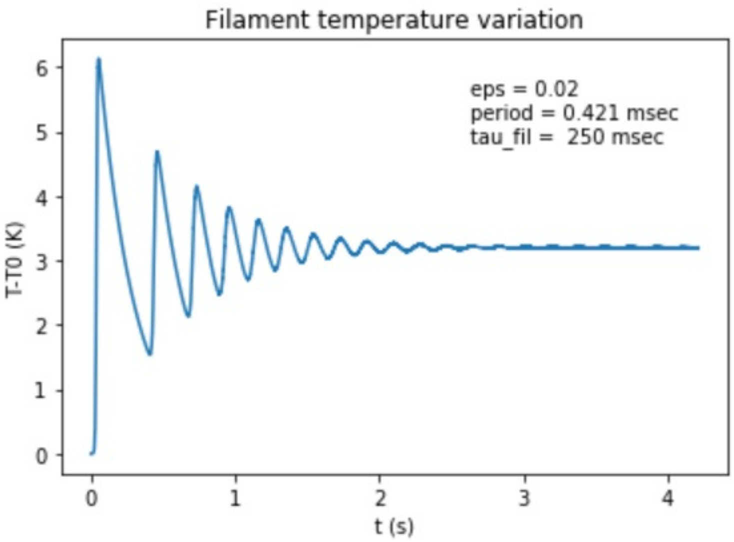
Radiation

Simulation over 10000 periods

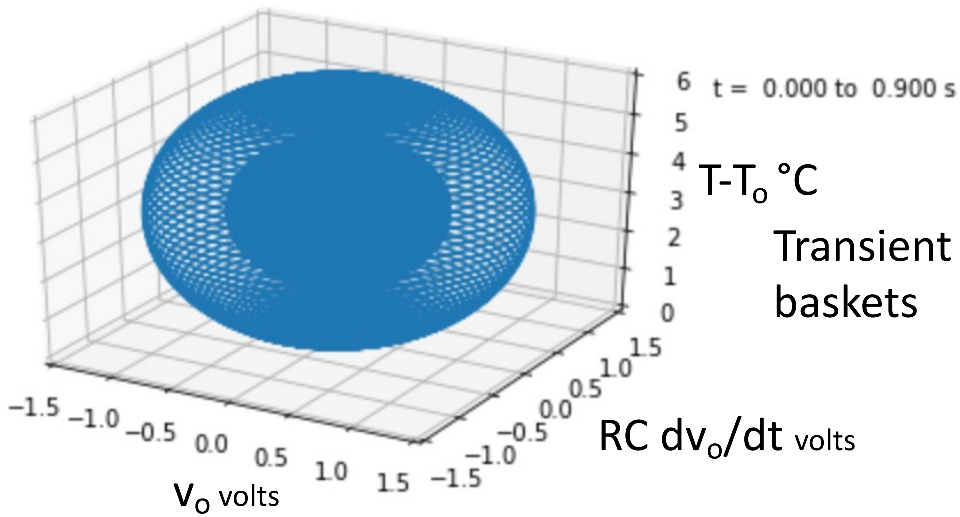
Voltage



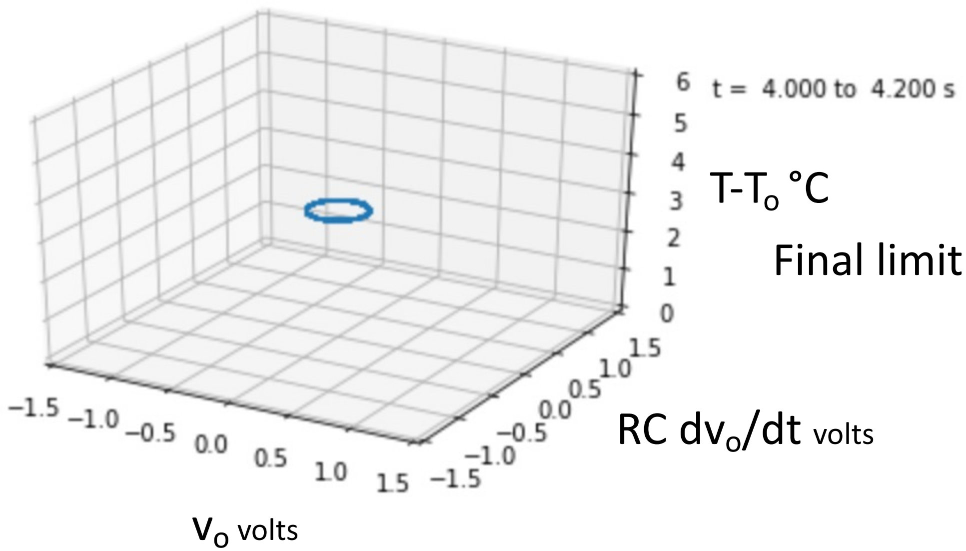
Filament temperature



3D phase portrait: *Matryoshka baskets*



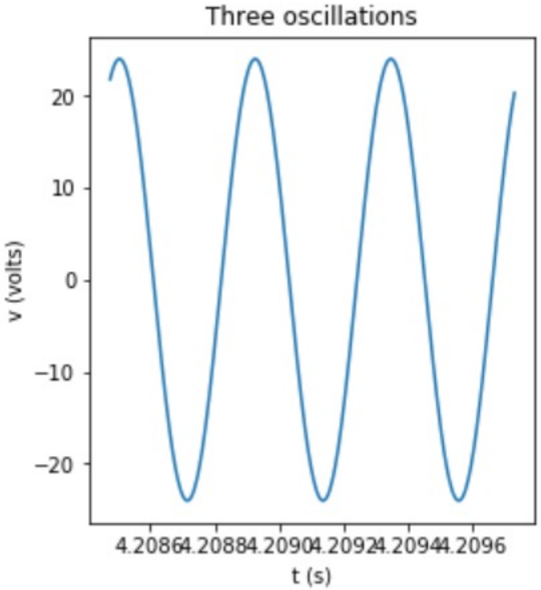
Transient nested baskets



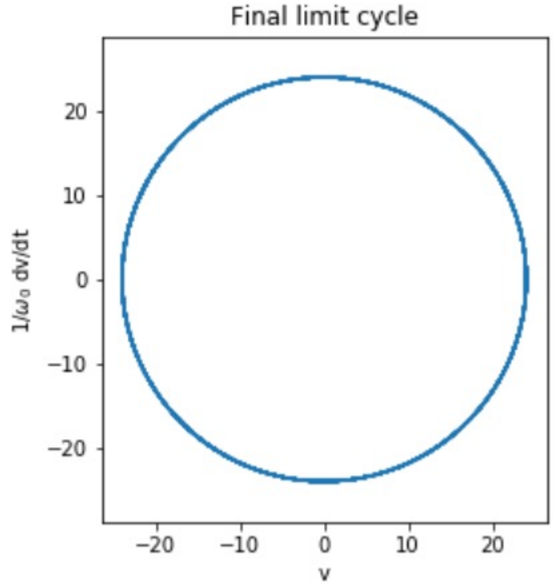
Final limit cycle

Signal after transients have settled: a *limit cycle.*

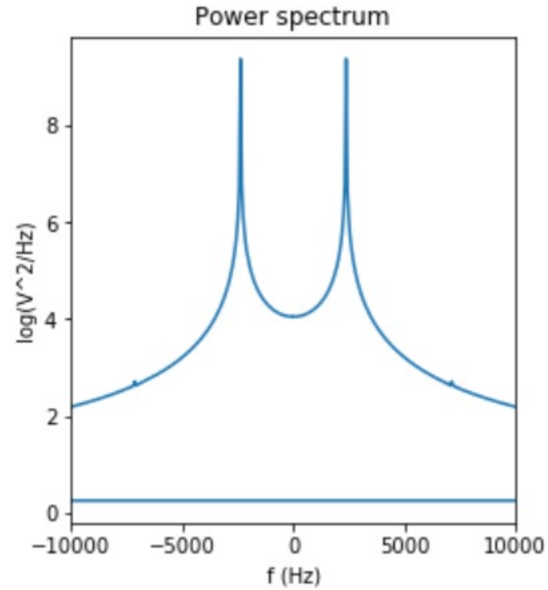
Time series



Phase portrait



Power spectrum



Long-time behavior

$$T - T_0 \approx \frac{1}{9R_{b0}\mathrm{K}_{eff}} \frac{1}{2} V^2 \qquad \text{where} \qquad V^2 = v^2 + w^2.$$

The dynamical system reduces to the Rayleigh – van der Pol equations

$$\frac{dv}{dt} = \omega_0 w.$$
$$\frac{dw}{dt} = -\omega_0 v + \left(\epsilon - \frac{1}{v_*^2}(v^2 + w^2)\right) \omega_0 w.$$

Not the van der Pol equations as is commonly assumed!

Transform to radial coordinates

$$v = r\cos\theta.$$
$$w = r\sin\theta.$$

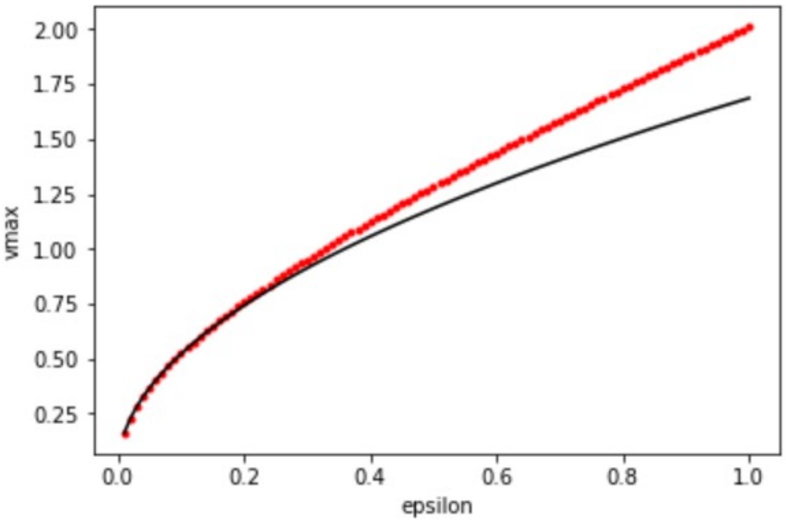
Radial form of Rayleigh – van der Pol equations

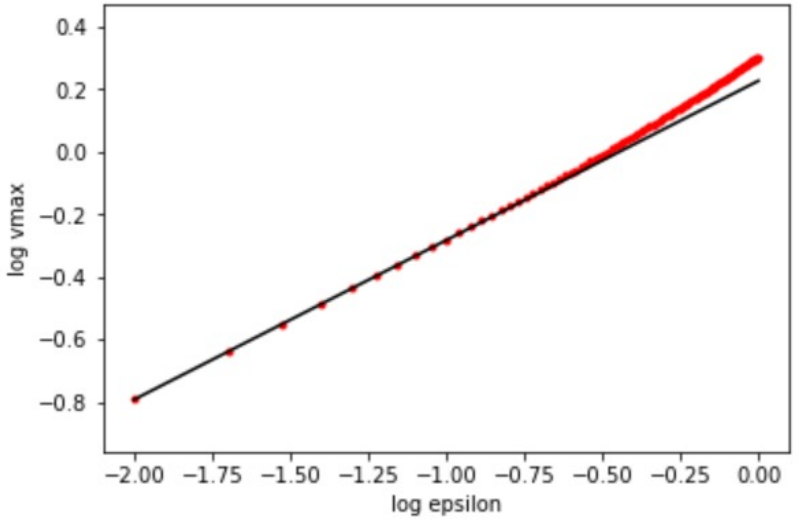
$$\frac{dr}{dt} = \left(\epsilon - \frac{r^2}{v_*^2}\right) \omega_0 r \sin^2\theta.$$
$$\frac{d\theta}{dt} = -\omega_0 + \left(\epsilon - \frac{r^2}{v_*^2}\right) \omega_0 \cos\theta \sin\theta.$$

Solution

$$r = v_* \sqrt{\epsilon}.$$
$$\theta = -\omega_0 t.$$

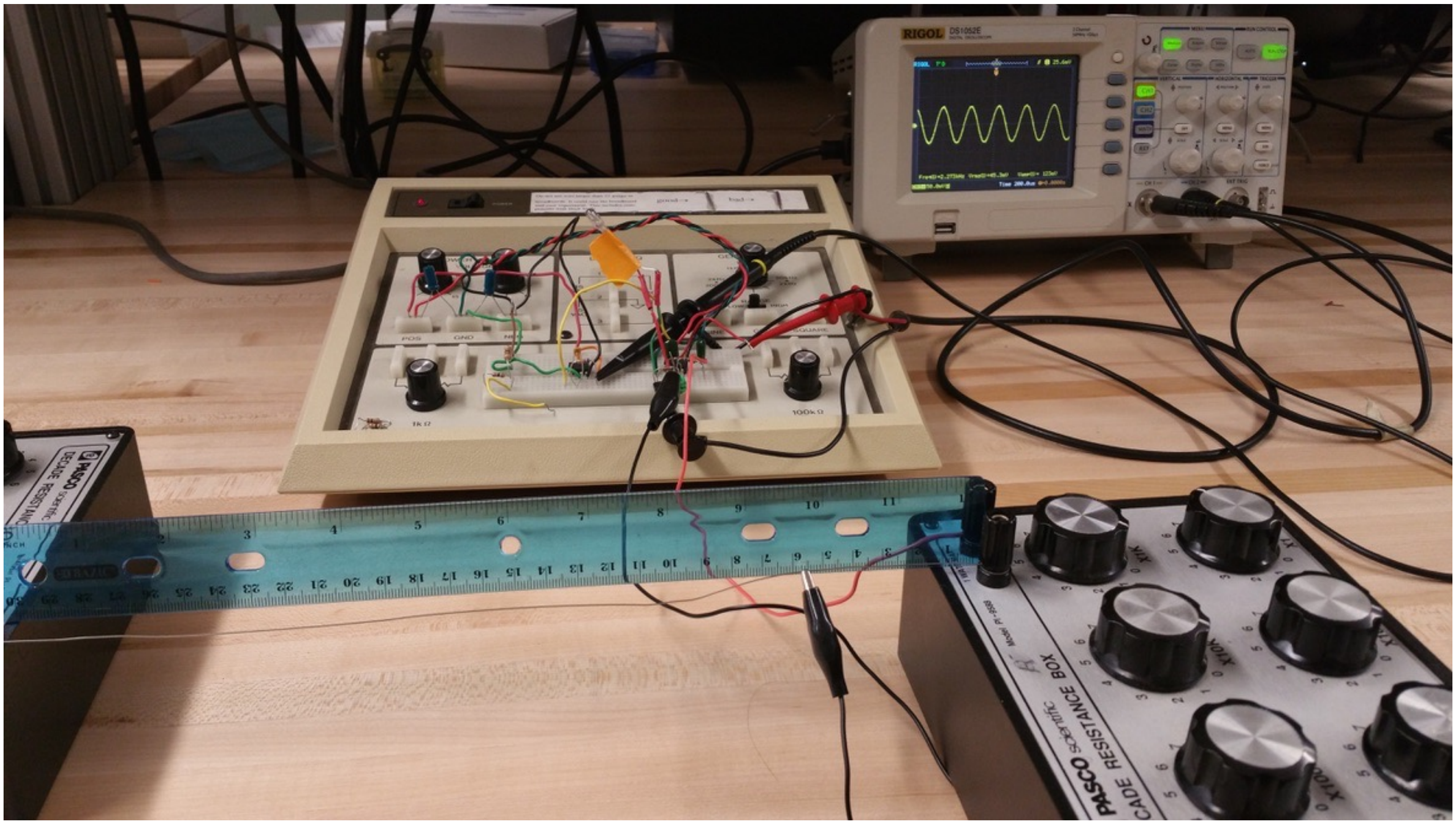
Growth of amplitude with parameter $\varepsilon = R_a/R_b - 2$



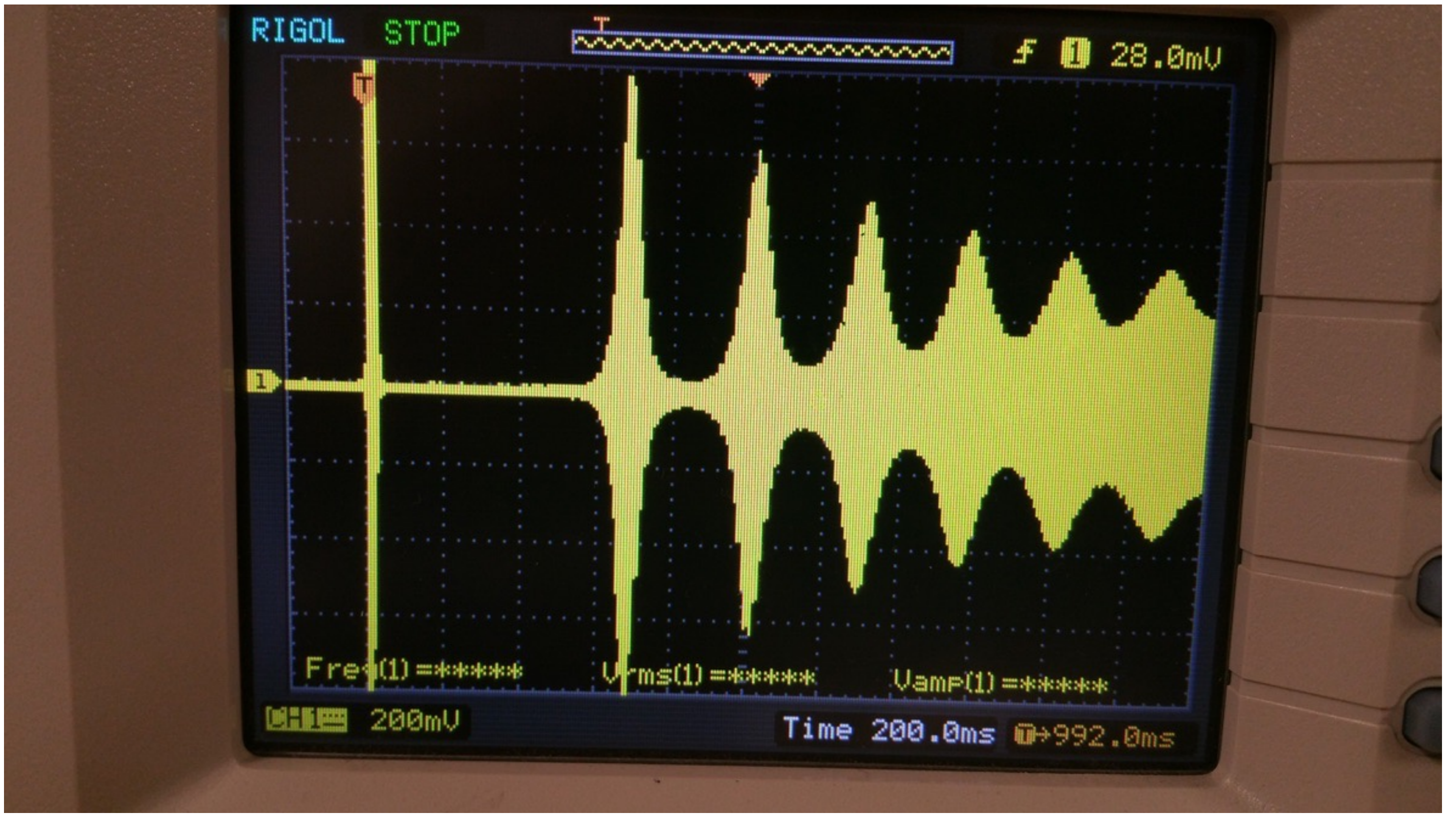


Initial dependence ~ $\varepsilon^{1/2}$

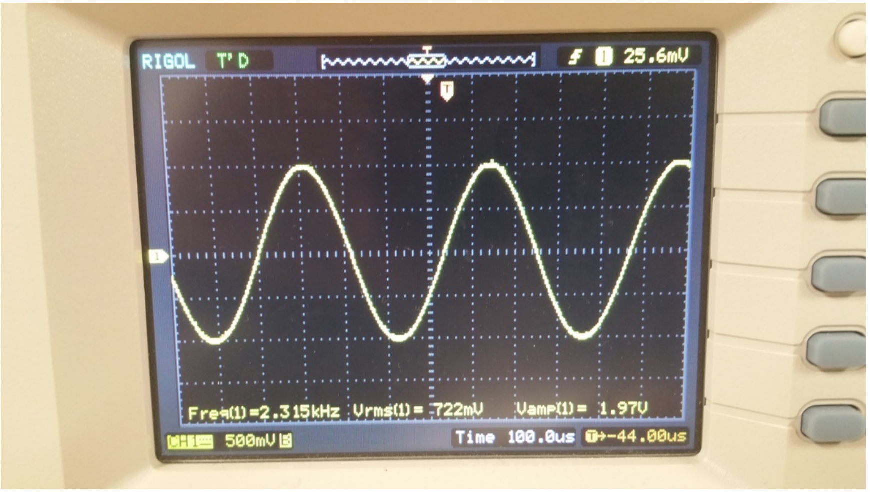
Experimental setup

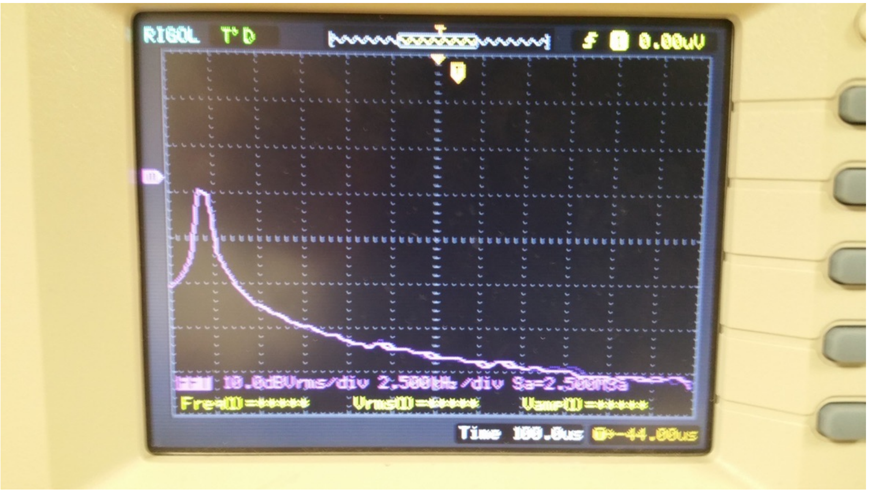


Transient squegging

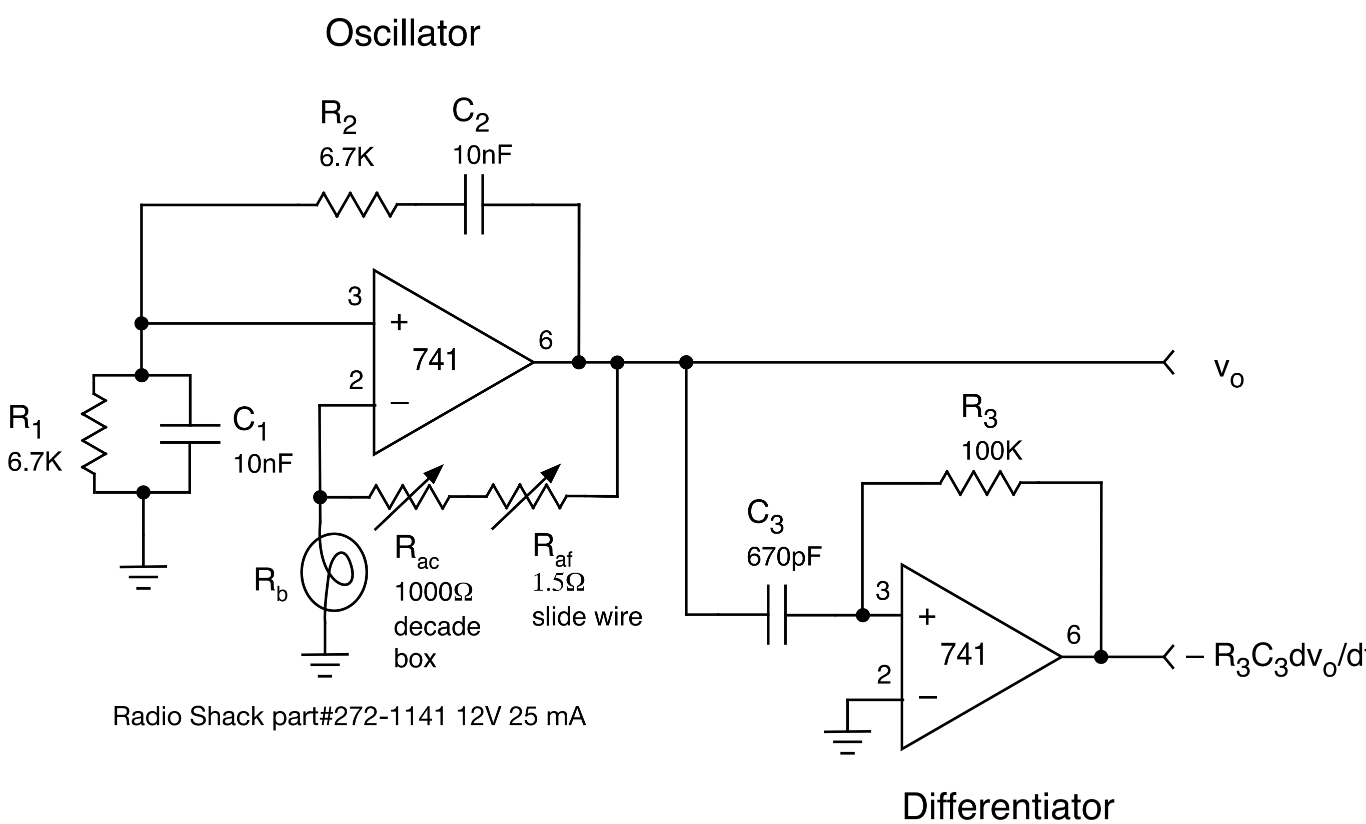


Long-term finite amplitude signal and its spectrum

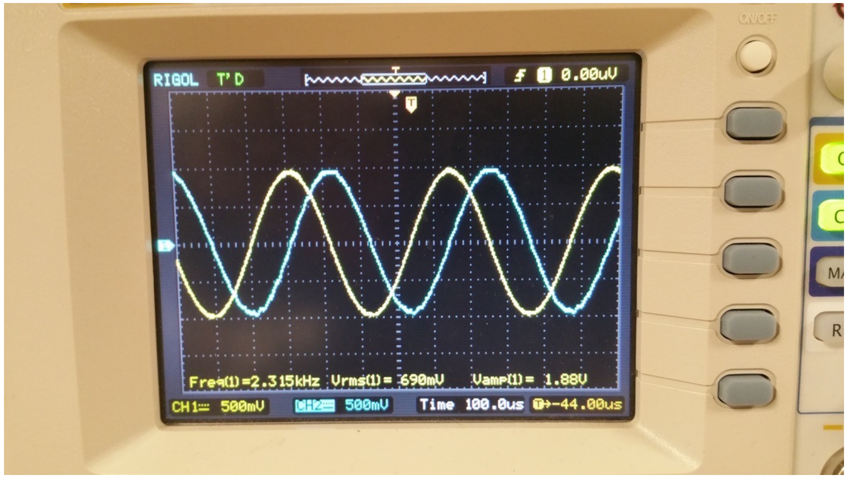


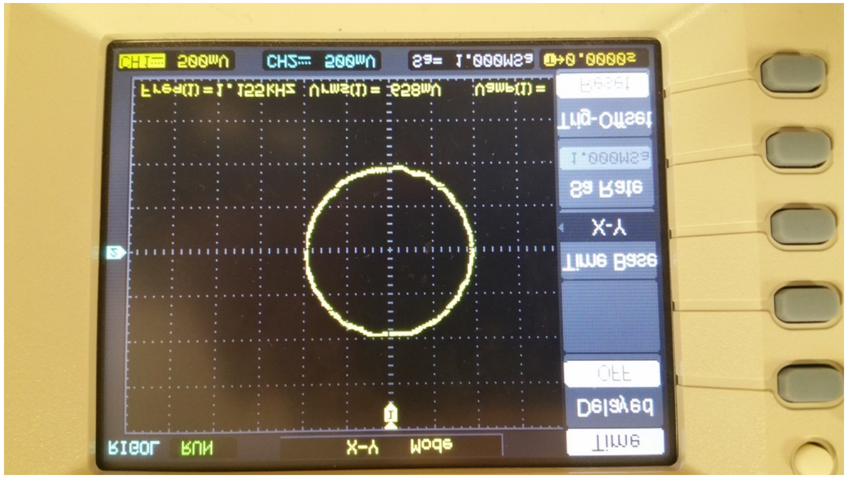


Include a differentiator

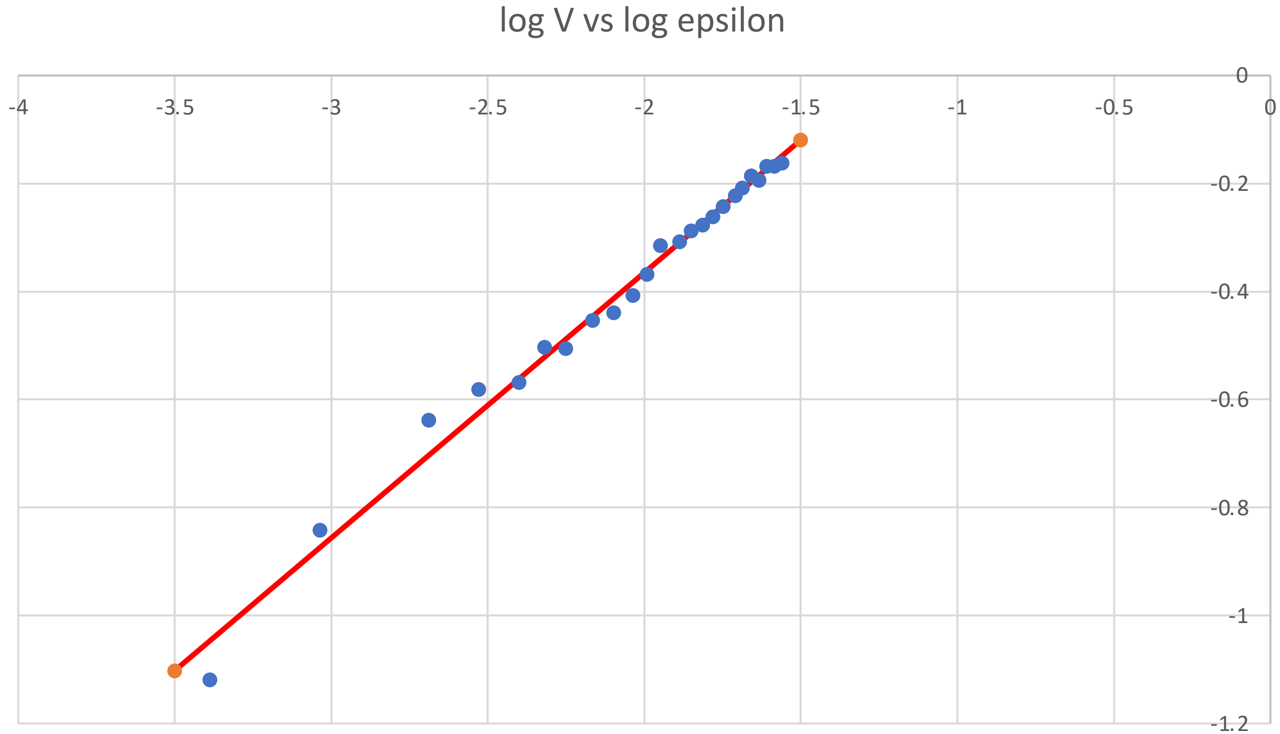


Time signals and phase portrait (xy mode)





Amplitude growth with parameter ε … needed <0.1 Ω resolution



Slope of 0.491 ± 0.011

Resistance range 107.19 Ω to 108.67 Ω

Topics for research & innovation

- The strange occurrence of persistent squegging when using the “wrong” type of capacitor
- Enhanced sensitivity to noise near the bifurcation: possible use for predicting changes.
- Marginal oscillator detectors, including models of the inner ear.
- Spontaneous changes in phases in coupled-oscillator systems, with applications to actuator systems and robotics.

Conclusions

- The Wien-bridge oscillator experiment develops important insights into the nonlinear dynamics of oscillators:
  - Bifurcations
  - Limit cycles
  - Scaling laws
- Laboratory work and simulation combine to enhance skills in
  - Circuit construction and measurement
  - Numerical modeling of physical systems (electronic & thermal)
- This equips students with a foundation for research and technical innovation.